Analysis of Penetrative Stellar Convection Using Singular Value Decomposition

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Objectives

In this research we seek to:

- Develop a novel approach toward characterizing steady-state and transient behavior in fluid flows
- Design and implement an efficient and scalable algorithm for such an analysis.
- Calculate the initial results of such an analysis on a well-defined system of physical interest.

Introduction

In the regimes of astrophysics and high-energy density physics, systems under study often exhibit incredibly complicated fluid flow patterns that arise from a variety of physical processes, with the relative importance of such processes often being shrouded due to the inherent nonlinearity of the system. In order to work around such issues, we have developed a method to describe the state of such a system as the linear combination of time-invariant empirical eigenfunctions, with associated time-dependent coefficients. Analysis of such eigenfunctions and temporal coefficients allows us to gain insight into which processes dominate throughout the time evolution of the fluid simulation. Further, such analysis allows us to characterize transients in the system where the time-invariant eigenfunctions fail to accurately approximate the solution.

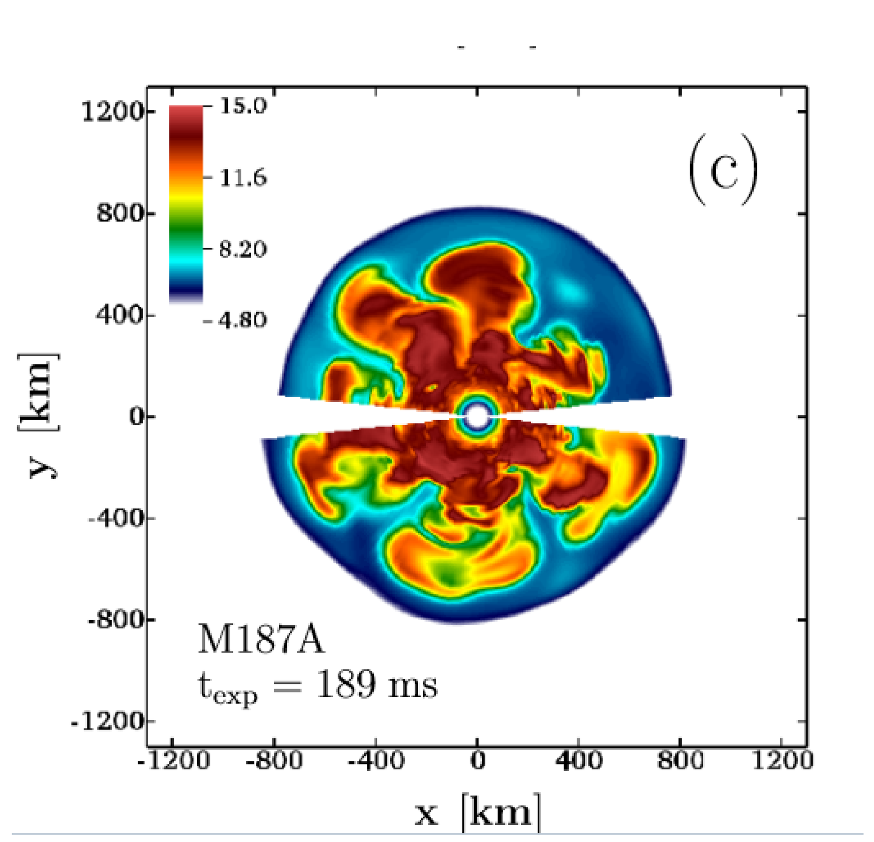

Figure 1: Explosion time entropy distribution in a core collapse supernova simulation (Handy, Plewa, & Odrzywołek 2013)

The Hurlburt Problem

Before development of our analysis tool, we must design a setup for our numerical experiments in which the interplay of the various physical processes occurs in a well understood manner. To this end, we adopt the model first proposed by Hurlburt et al. in 1986. In such a setup, two layers of fluid stable against convection šandwichä layer that is convectively unstable. The initial profiles of the state variables temperature, density and pressure are given as polytropes as a function of depth in the form

$$T = T_i + \frac{z - z_i}{K_i}, \quad \frac{\rho}{\rho_i} = \left(\frac{T}{T_i}\right)^{m_i}, \quad \frac{P}{P_i} = \left(\frac{T}{T_i}\right)^{m_i+1} \quad (1)$$

Where K_i is the coefficient of thermal conductivity for each layer i, judiciously selected such that the Schwarzschild criterion for convective instability,

$$\frac{F_T}{K_i} > \frac{g}{C_p} \quad (2)$$

is satisfied only for the central layer, where C_p is the specific heat of the fluid at constant pressure, F_T is the thermal flux at the bottom of each fluid layer, and g is the uniform downward gravitation acceleration acting on the fluid.

These initial conditions are implemented within the FLASH Astrophysics code, which then evolves the system of extended Euler equations until the system relaxes and and a steady state solution is acheived.

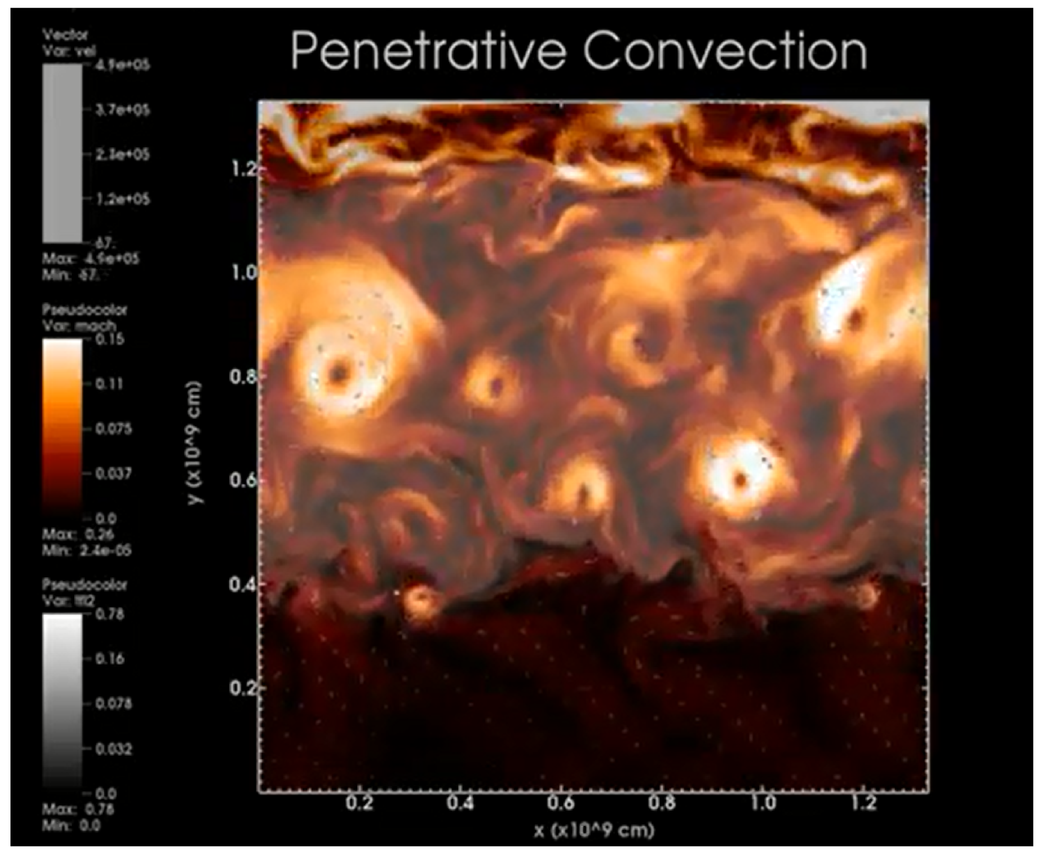

Figure 2: Pre-Steady State Evolution of the Hurlburt Setup

Singular Value Decomposition

In seeking to describe the solution of nonlinear equations as the linear combination of a finite number of modes, we must first select basis functions on which to project our solution. A great deal of work has previously gone into this problem within the fields of model reduction, and it has been shown that using a basis of empirical eigenfunctions derived from the solution matrix provides the optimal basis, a method known as Proper Orthogonal Decomposition.

To compute this basis, we utilize the method of snapshots as proposed by Sirovich , begining with a mxn matrix D of our solution data, where each column is the linearized solution matrix at each of n snapshots. We then mean center the data, by subtracting the time-averaged value of each row from each element within that row such that

$$\tilde{D} = D - \bar{D} \quad (3)$$

From $\tilde{D}$, we can make substantially computational savings by calculating the SVD indirectly, using the covariance matrix

$$C = \tilde{D}\tilde{D}^T \quad (4)$$

Yielding, an $n \times n$ array, reducing the problem size immensely, as $m >> n$. We then compute the right eigenvectors of $C; U$, which can be used to calculate the optimal orthogonal basis on which to project our solution; V^T, through the following equation.

$$U^T C = \Sigma V^T \quad (5)$$

We can then approximate each column of C with the linear combination of dominant eigenmodes of V^T (ranked according to the magnitude of the corresponding eigenvalue) along with least squares coefficients in the form

$$C_i \approx \alpha_i V^T \quad (6)$$

SVD Derived Basis Functions

A subset of the computed basis functions of one of our numerical experiments is presented below, followed by the eigenvalue spectrum for all computed eigenmodes.

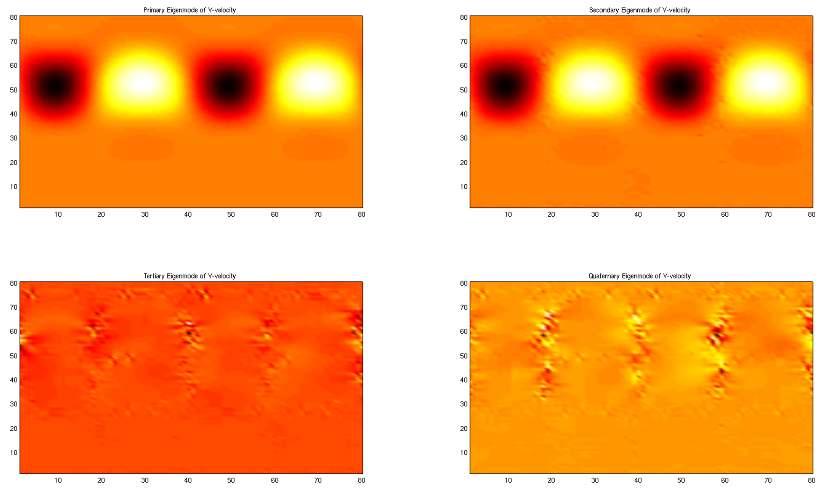

Figure 3: Four dominant eigenmodes of our Hurlburt simulation

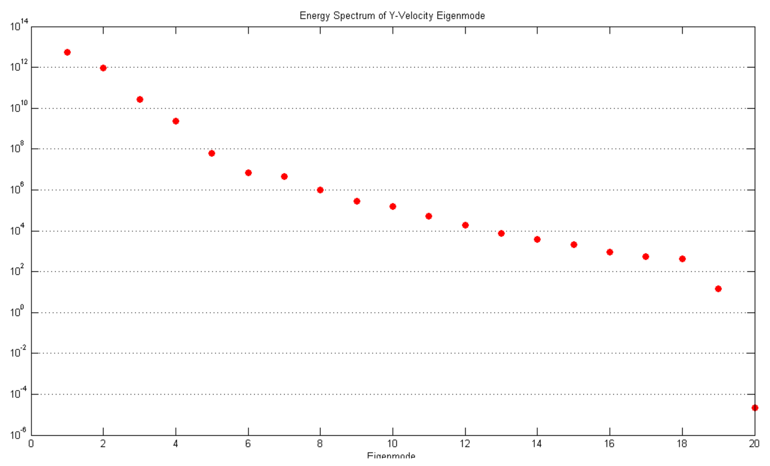

Figure 4: Energies associated with computed eigenmodes

Here we see that the vast majority of the information within the system can be ascribed to one of the first few eigenmodes, allowing us to truncate the linear expansion after as little as two terms, yet still describe the solution from which these eigenmodes were computed in an accurate manner.

Conclusions and Future Work

In this work, we have demonstrated our technique for decomposing solutions of nonlinear systems into a linear combination of empirical eigenfunctions. Our work thus far has been limited to trial systems or unperturbed models, but the full potential of such a method of analysis will only be realized once perturbed fluid models are characterized in such a manner, with our method allowing us to identify any subtle transients phases via analysis of of our calculated time-dependent coefficients.