Numerical Analysis of Finite Element Method for Nonlocal Obstacle Problems

Qingguang Guan Advisor: Max Gunzburger

Florida State University

Abstract

For classical obstacle problem, the obstacle function ψ should be a smooth function, however, we considered the nonlocal obstacle problem which can allow ψ to be a function with jump discontinuities. The Laplacian operator in classical obstacle problem is replaced by the nonlocal operator, for which the fractional Laplacian is a special case. We considered the problem of minimizing an energy functional with the constraint $u \geq \psi$, where u is the solution of the problem. In a proper space, we proved the existence and uniqueness of the solution. A finite element method is applied to solve the minimization problem. The convergence of the numerical solution to the exact solution was proved. Also, we gave some numerical results to verify our theorems.

Introduction

In general, the obstacle function ψ should be a smooth (at least continuous) function, and the solution u is continuous and possesses Lipschitz continuous first derivatives. However, what if ψ is discontinuous? Obviously, it is difficult for the classical obstacle problem to deal with. Since the local problem is the limit of the nonlocal problem, we can develop the nonlocal obstacle problem to overcome it. For the nonlocal obstacle problem we can set ψ as a function with discontinuities or with less regularity. There are two mainly cases.

We define

$$\gamma(x,y) = \begin{cases} \frac{c_\gamma}{|y-x|^{n+2s}}, & |y-x| \leq \delta, \\ 0, & |y-x| > \delta. \end{cases}$$

Case 1, $s \in (0,1),\ \delta > 0,\ c_\gamma > 0,\ x, y \in \mathbb{R}^n$.

Case 2, $s \in [-n/2, 0),\ \delta > 0,\ c_\gamma > 0,\ x, y \in \mathbb{R}^n$.

For both cases, in proper spaces, we need to find $u \geq \psi$, such that

$$I[u] = \min_{v \geq \psi} I[v], \tag{1}$$

where $I[v] := \frac{1}{2}\int_\Omega \int_\Omega \gamma(x,y)(v(x) - v(y))^2\, dydx - \int_\Omega vf\, dx$.

Case 1: Existence and Uniqueness

Let $\Omega_I \subset \mathbb{R}^n$ be a bounded open domain with piecewise smooth boundary. The boundary domain is: $\Omega_B := \{y \in \mathbb{R}^n \setminus \Omega_I \text{ such that } \gamma(x,y) \neq 0 \text{ for } x \in \Omega_I\}$, and let $\Omega = \Omega_I \cup \Omega_B$.

We define the semi-norm of u and the associated spaces as:

$$|||u||| := \left(\int_\Omega \int_\Omega \gamma(x,y)(u(x) - u(y))^2 dydx \right)^{1/2}$$

$$V_c^s(\Omega) := \{u \in L^2(\Omega) : |||u||| < \infty \text{ and } u(x) = 0, x \in \Omega_B\},$$

$V_{c,0}^s(\Omega)$ is the closure of $C_0^\infty(\Omega_I)$ in $V_c^s(\Omega)$.

Let $\psi \in V_c^s(\Omega)$, we define $\mathcal{A} := \{u \in V_{c,0}^s(\Omega) : u \geq \psi\}$.

Lemma 1. *$\mathcal{A}$ is closed and convex.*

Our goal is to seek $u \in \mathcal{A}$ such that (1) is satisfied.

Lemma 2. *$I[\cdot]$ is weakly lower semicontinuous on $V_{c,0}^s(\Omega)$, which is $I[v] \leq \liminf_{k\to\infty} I[v_k]$ whenever $v_k \rightharpoonup v$, weakly in $V_{c,0}^s(\Omega)$.*

Lemma 3. *Suppose $v, \psi \in V_c^s, v \geq \psi$ and $\lim_{n\to\infty} \psi_n \to \psi$ strongly in V_c^s, $\lim_{n\to\infty} v_n \to v$ strongly in V_c^s, then we have $\lim_{n\to\infty} \max(v_n, \psi_n) \to v$ strongly in V_c^s.*

Then we define

$$a(u,v) = \int_\Omega \int_\Omega \gamma(x,y)(u(x) - u(y))(v(x) - v(y))dxdy,$$

$$(f,v) = \int_\Omega f v dx.$$

Theorem 1. *$u \in \mathcal{A}$ solves (1) if and only if it solves*

$$a(u, v-u) \geq (f, v-u),\ \forall v \in \mathcal{A}. \tag{2}$$

Theorem 2. *The solution of (2) exists and is unique.*

Case 1: Approximation

Suppose that we are given a parameter h converging to 0, and a family $\{V_h\}_h$ of closed subspaces of $V_{c,0}^s$. Also, there is a family $\{\mathcal{A}_h\}_h$ of closed convex nonempty subsets of $V_{c,0}^s$ with $\mathcal{A}_h \subset V_h$, such that $\{\mathcal{A}_h\}_h$ satisfies the following two conditions:

(i), If $v_h \in \mathcal{A}_h$, and $\{v_h\}_h$ is bounded in $V_{c,0}^s$, then the weak cluster points of $\{v_h\}_h$ belong to $\mathcal{A}$.

(ii), There exists $\chi \subset V_{c,0}^s, \bar{\chi} = \mathcal{A}$ and $r_h : \chi \to \mathcal{A}_h$ such that $\lim_{h\to 0} r_h v = v$ strongly in $V_{c,0}^s$, $\forall v \in \chi$.

The problem (2) is approximated by

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h), \quad \forall v_h \in \mathcal{A}_h, u_h \in \mathcal{A}_h. \tag{3}$$

Theorem 3. *(3) has a unique solution. With the above assumptions on $\mathcal{A}$ and $\{\mathcal{A}_h\}_h$, we have*

$$\lim_{h\to 0} |||u_h - u||| = 0,$$

with u_h the solution of (3) and u the solution of (2).

We define $\mathfrak{D} := \{v|_{\Omega_I} \in C_0^\infty(\Omega_I), \text{ and } v|_{\Omega_B} = 0\}$.

Theorem 4. *Let $\psi \in V_c^s(\Omega)$, if there exist functions $\{\psi_n\}_n \in V_c^s$ and $\psi_n|_{\Omega_I} \in C(\Omega_I)$, also, $\psi_n \leq 0$ in a neighborhood of $\partial\Omega_I$, and $\psi_n \geq \psi$, $\psi_n \to \psi$, $n \to \infty$, strongly in V_c^s, then we have $\overline{\mathfrak{D} \cap \mathcal{A}} = \mathcal{A}$.*

From now on, we restrict ourselves to continuous piecewise linear finite element approximations. We define the regular triangulation of Ω_I as $\mathcal{T}_h$. $\Sigma_h := \{p \in \bar{\Omega}_I, p \text{ is a vertex of } T \in \mathcal{T}_h\}$. The space $V_{c,0}^s$ is approximated by the family of subspaces $\{V_h\}_h$, where

$$V_h := \{v_h \in C^0(\Omega), v_h|_{\Omega_B} = 0 \text{ and } v_h|_T \in P_1, \forall T \in \mathcal{T}_h\}.$$

Thus we can get the basis functions $\{\phi_p\}$, such that $v_h = \sum v_h(p)\phi_p$.

Case 1: Assumptions and Examples of ψ

Assumption 1. ψ satisfies the conditions in Theorem 4.

Assumption 2. $\lim_{h\to 0} \psi_h = \psi$ strongly in $L^2(\Omega)$, where $\psi_h(p) = \psi(p),\ \psi_h \in C^0(\Omega)$, and $\psi_h|_T \in P_1$.

Then we approximate $\mathcal{A}$ by $\mathcal{A}_h = \{v_h \in V_h, v_h(p) \geq \psi_h(p), \forall p \in \Sigma_h\}$.

Example 1. In 1-D, we define $\Omega_I = (a,b)$, and set the sub-intervals: $I_1 = (a, a_1), I_2 = [a_1, a_2), \cdots, I_m = [a_{m-1}, b)$, where m is a finite number. $\psi \in \{\psi | \ \psi|_{\bar{I}_i} \in C^1(\bar{I}_i),\ i = 1, \cdots, m\}$ and $\psi \leq 0$ in a neighborhood of $\partial\Omega_I$.

Example 2. In 2-D, suppose there are two circles in Ω_I dividing it into three parts. Ω_1, Ω_2 are in the circles, and $d(\Omega_1, \Omega_2) > 0$, we have $\psi = 1, \forall x \in \Omega_1 \cup \Omega_2$, and $\psi = 0$ elsewhere.

Example 3. $\psi \in \{\psi | \ \psi|_{\Omega_I} \in H_0^1(\Omega_I) \cap C_0^0(\Omega_I), \psi|_{\Omega_B} = 0\}$.

Remark 1. *For $0 < s < \frac{1}{2}$, Example 1 and 2 satisfy the two assumptions of ψ. For $0 < s < 1$, Example 3 satisfies the two assumptions of ψ.*

Conclusions

- For Case 1, if ψ satisfies the two assumptions, with the definition of $\mathcal{A}_h$, then $\{\mathcal{A}_h\}_h$ satisfies conditions (i) and (ii), so that we have Theorem 3.
- For Case 2, we define $V_c(\Omega) := \{u : \|u\| < \infty \text{ and } u(x) = 0, x \in \Omega_B\}$, where $\|\cdot\|$ is the $L_2(\Omega)$ norm, then let $\psi \in V_c(\Omega)$ as Example 1-3.
  We define $\mathbb{A} := \{u \in V_c(\Omega) : u \geq \psi\}$.

  Theorem 5. *The solution $u \in \mathbb{A}$ of (1) exists and is unique.*

  Then we approximate $\mathbb{A}$ by $\mathbb{A}_h = \{v_h \in V_h, v_h(p) \geq \psi_h(p), \forall p \in \Sigma_h\}$.
  The problem (1) is approximated by

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h), \quad \forall v_h \in \mathbb{A}_h, u_h \in \mathbb{A}_h. \tag{4}$$

  Theorem 6. *(4) has a unique solution. With the above assumptions on $\mathbb{A}$ and $\{\mathbb{A}_h\}_h$, we have*

$$\lim_{h\to 0} \|u_h - u\| = 0,$$

  with u_h the solution of (4) and u the solution of (1).

Numerical Results in 1-D

a). Here let $s = -0.05$, $h = 1/257$, $c_\gamma = \frac{2-2s}{\delta^{2-2s}}$, $f = 0$, and ψ be the discontinuous obstacle as

$$\psi = \begin{cases} 2 - 8x & \text{if } \frac{1}{8} < x \leq \frac{1}{4}; \\ 8x - 6 & \text{if } \frac{3}{4} < x \leq \frac{7}{8}; \\ 0 & \text{elsewhere.} \end{cases}$$

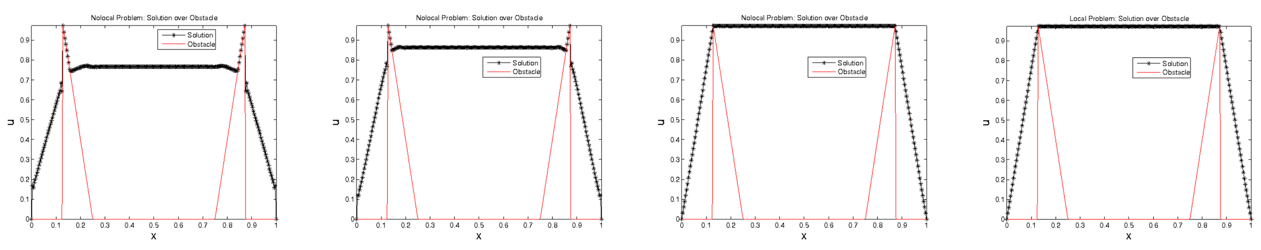

Figure 1: From left to right: $\delta = 0.1$ (nonlocal) and $\delta = 0.05$ (nonlocal), $\delta = 0.001$ (nonlocal), local.

b). Here let $s = -0.45$, $\delta = 0.1$, $c_\gamma = \frac{2-2s}{\delta^{2-2s}}$, $f = 0$, and ψ be the discontinuous obstacle as

$$\psi = \begin{cases} 1 & \text{if } \frac{1}{4} < x \leq \frac{3}{4}; \\ 0 & \text{elsewhere.} \end{cases}$$

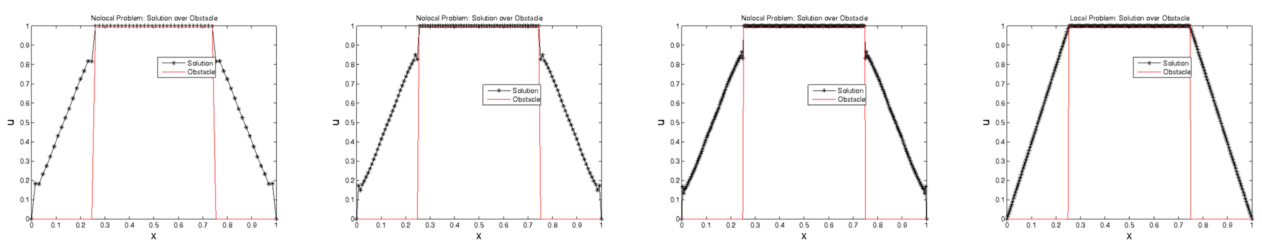

Figure 2: From left to right: $h = 1/65$ (nonlocal), $h = 1/129$ (nonlocal), $h = 1/257$ (nonlocal), and $h = 1/257$ (local).

c). Here let $s = 1/4$, $\delta = 2$, $c_\gamma = \frac{2-2s}{\delta^{2-2s}}$, $f = 0$, and ψ is the same as the one in a).

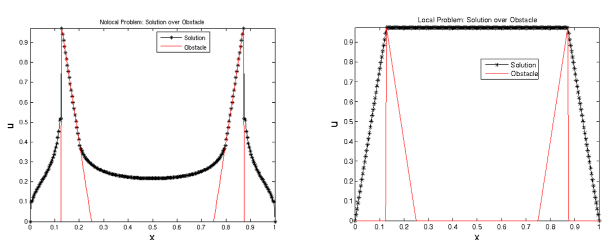

Figure 3: From left to right: $h = 1/257$ (nonlocal), and $h = 1/257$ (local).

References

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